ARULMIGU PALANIANDAVAR ARTS COLLEGE FOR WOMEN,

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DEPARTMENT OF MATHEMATICS

LEARNING RSOURCES

SUBJECT: ORDINARY DIFFERENTIAL EQUATIONS

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11/8/23

ODE NOTES : Unit 1 : Linear Equation with Constant Coefficients

Def: A linear differential Equation of order n with constant coefficients is an equation of the form

$$a_0 y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \dots + a_n y = b(x)$$

where $a_0 \neq 0$, $a_1, \dots a_n$ are complex constants and $b(x)$ is some complex valued function on an interval I. We take $a_0 = 1$, then the above equation can be written in the form

$$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \dots + a_n y = b(x)$$

ie, $L(y) = y^{(n)} + a_1 y^{(n-1)} + \dots + a_n y = b(x)$.

If $b(x) = 0$ for all x in I, then the equation is said be a homogeneous equation, whereas $L(y) = b(x)$ is called a non homogeneous equation. If $b(x) \neq 0$ for some x in I.

The second order homogenous Equations

The second order homogeneous linear differential Equation is

$$L(y) = y'' + a_1 y' + a_2 y = 0$$ where a_1 and a_2 are constant.

The solution of the first order equation $y' + ay = 0$ with constant coefficients is e^{-ax}. The constant $-a$ in this solution is the solution of the equation $r + a = 0$

Since differentiating an exponential e^{rx} any number of times where r is a constant, always yields a constant times e^{rx}.

$$L(e^{rx}) = (r^2 + a_1 r + a_2)\, e^{rx}$$

and e^{rx} will be a solution of $L(y) = 0$ $L(e^{rx}) = 0$ if r satisfies $r^2 + a_1 r + a_2 = 0$

we let $p(r) = r^2 + a_1 r + a_2$ is called the characteristic polynomial of L.

$p(r)$ can be obtained from $L(y)$ by replacing $y^{(k)}$ everywhere by r^k where we use the convention that the zeroth derivative of the y, $y^{(0)}$ is y itself and that $r^0 = 1$. From the fundamental theorem of Algebra we know that the polynomial p always has two complex roots r_1, r_2 (which may be real). If $r_1 \neq r_2$ we see that $e^{r_1 x}$ and $e^{r_2 x}$ are two distinct solution of $L(y) = 0$.

$$L(e^{rx}) = p(r) e^{rx}$$

for all r and x. we recall that if r_1 is a repeated root of p then not only $p(r_1) = 0$ but

$$p'(r_1) = 0.$$

$$\frac{\partial}{\partial r} L(e^{rx}) = L\left(\frac{\partial}{\partial r} e^{rx}\right) = L(x e^{rx})$$

$$L(x e^{rx}) = \{p'(r) + x p(r)\} e^{rx}$$

Theorem 1: Let a_1, a_2 be constants and consider the equation $L(y) = y'' + a_1 y' + a_2 y = 0$

If r_1, r_2 are distinct roots of the characteristic polynomial p where $p(r) = r^2 + a_1 r + a_2$ then the functions ϕ_1, ϕ_2 defined by

$$\phi_1(x) = e^{r_1 x}, \quad \phi_2(x) = e^{r_2 x}.$$

are solution of $L(y) = 0$. If r_1 is a repeated root of p then the functions ϕ_1, ϕ_2 defined by

$$\phi_1(x) = e^{r_1 x}, \quad \phi_2(x) = x e^{r_1 x}$$ are solutions of $L(y) = 0$

If ϕ_1, ϕ_2 are any two solution of $L(y) = 0$ and c_1, c_2 are two constants, then the function $\phi = c_1 \phi_1 + c_2 \phi_2$ is also a solution of $L(y) = 0$

$$L(\phi) = (c_1 \phi_1 + c_2 \phi_2)'' + a_1 (c_1 \phi_1 + c_2 \phi_2)' + a_2 (c_1 \phi_1 + c_2 \phi_2)$$

$$= c_1 \phi_1'' + c_2 \phi_2'' + c_1 a_1 \phi_1' + c_2 a_1 \phi_2' + c_2 a_2 \phi_1 + c_2 a_2 \phi_2$$

$$= c_1 L(\phi_1) + c_2 L(\phi_2) = 0$$

The function ϕ which is zero for all x is also a solution the trivial solution of $L(y) = 0$

Example

1) Solve the equation $y'' + y' - 2y = 0$

The characteristic polynomial is

$$p(r) = r^2 + r - 2$$

$p(r) = 0 \Rightarrow r^2 + r - 2 = 0$

$\Rightarrow r = -2, 1$

$\therefore$ Solution, $\phi(x) = \underline{\underline{c_1 e^{-2x} + c_2 e^{x}}}$

where c_1, c_2 are constant.

Ex 2) Solve $y'' + \omega^2 y = 0$

The characteristic poly is $p(r) = r^2 + \omega^2$.

$p(r) = 0 \Rightarrow r^2 + \omega^2 = 0$

$\Rightarrow r^2 = -\omega^2 = \omega^2 i^2$

$r = i\omega, -i\omega$

Solution $\phi = \underline{\underline{c_1 e^{i\omega x} + c_2 e^{-i\omega x}}}$

where c_1, c_2 are two constants. We see that $c_1 = 1/2$, $c_2 = 1/2$

$\cos \omega x$ is a solution. $c_1 = 1/2i$, $c_2 = -1/2i$

$\sin \omega x$ is a solution.

Exercise

I. Find all solution of the following equation

1. $y'' - 4y = 0$

Given $y'' - 4y = 0$ — ①

The characteristic polynomial is $p(r) = r^2 - 4 = 0$

ie $r^2 - 4 = 0$

$\Rightarrow r^2 = 4$

$\Rightarrow r = \pm 2$

∴ The soln ϕ has of the form

$\phi(x) = c_1 e^{2x} + c_2 e^{-2x}$

b) $3y'' + 2y' = 0$

Given $3y'' + 2y' = 0$ — ①

The characteristic polynomial, $p(r) = 3r^2 + 2r = 0$

ie, $r(3r+2) = 0$

$r = 0, \quad 3r+2 = 0$

ie, $3r = -2$,

$r = -2/3$ or $r = 0$

∴ The solution $\phi = c_1 e^{0x} + c_2 e^{-2/3 x}$

$= c_1 + c_2 e^{-2/3 x}$

c) $y'' + 16y = 0$

Given $y'' + 16y = 0$

The characteristic polynomial is

$p(r) = r^2 + 16$

ie, $r^2 + 16 = 0$

$r^2 = -16$

$r^2 = -A^2 = \underline{\underline{\pm 4i}}$

$\therefore$ The solution $\phi = \underline{\underline{C_1 e^{4ix} + C_2 e^{-4ix}}}$

d) $y'' = 0$

Given $y'' = 0$

The characteristic polynomial

$p(r) = r^2 = 0$

$\therefore r = 0$ (twice)

The solution $\phi = C_1 e^{0x} + C_2 e^{0x} = \underline{\underline{C_1 + C_2}}$

e) $y'' + 2iy' + y = 0$

Given $y'' + 2y' + 2y = 0$

The characteristic polynomial

$p(r) = r^2 + 2ir + 1$

i.e, $r^2 + 2ir + 1 = 0$

$a = 1, \quad b = 2i, \quad c = 1$

$r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \dfrac{-2i \pm \sqrt{-4 - 4}}{2} = \dfrac{-2i \pm \sqrt{-8}}{2}$

$= \dfrac{-2i \pm 2i\sqrt{2}}{2}$

$= \underline{\underline{-i \pm i\sqrt{2}}}$

The solution ϕ has of the form,

$\phi(x) = C_1 \underline{e^{(\sqrt{2}-1)x} + C_2 e^{-(\sqrt{2}-1)x}}$

f) $y'' - 4y' + 5y = 0$

Given $y'' - 4y' + 5y = 0$

The characteristic polynomial is

$p(r) = r^2 - 4r + 5$

$$r^2 - 4r + 5 = 0$$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{4 \pm \sqrt{16-20}}{2}$$

$$= \frac{4 \pm \sqrt{-4}}{2} = \frac{4 \pm 2i}{2}$$

$$r = 2 \pm i$$

The soln ϕ has the forms

$$\phi(x) = c_1 e^{(2+i)x} + c_2 e^{(2-i)x}$$

$$\phi(x) = e^{2x}(c_1 \cos x + c_2 \sin x)$$

9) $y'' + (3i - 1)y' - 3iy = 0$

$$y'' + (3i-1)y' - 3iy = 0$$

The characteristic polynomial is

$$p(r) = r^2 + (3i - 1)r - 3i$$

$$p(r) = 0 \Rightarrow r^2 + (3i-1)r - 3i = 0$$

$$r = \frac{(3i-1) \pm \sqrt{(3i-1)^2 - 4 \times -3i}}{2}$$

$$= \frac{-3i + 1 \pm \sqrt{-9 - 2\,3i + 1 + 12i}}{2}$$

$$= \frac{-3i + 1 \pm \sqrt{9i^2 + 6i + 1}}{2}$$

$$= \frac{-3i + 1 \pm \sqrt{(3i+1)^2}}{2}$$

$$= \frac{-3i+1 \pm (3i+1)}{2} = \frac{-3i+1+3i+1}{2}, \frac{-3i+1-3i-1}{2}$$

$$= \frac{2}{2}, \frac{-6i}{2}$$

$$= 1, -3i$$

The soln, $\phi = \underline{c_1 e^{x} + c_2 e^{-3ix}}$

2. Consider the equation $y'' + y' - 6y = 0$
compute the soln ϕ satisfying $\phi(0) = 1, \phi'(0) = 0$

The characteristic polynomial is

$p(r) = r^2 + r - 6$

ie, $r^2 + r - 6 = 0$

$\Rightarrow (r+3)(r-2) = 0$

$\underline{r = -3, 2}$

∴ The solution ϕ has the form

$\phi(x) = c_1 e^{-3x} + c_2 e^{2x} \rightarrow (1)$

Given $\phi(0) = 1, \phi'(0) = 0$

$\phi'(x) = c_1 -3 e^{-3x} + 2 c_2 e^{2x}$

$\phi(0) = 1 \Rightarrow c_1 + c_2 = 1 \rightarrow (2)$

$\phi'(0) = 0 \Rightarrow \phi'(0) = -3c_1 + 2c_2 = 0 \rightarrow (3)$

$$\begin{array}{r} 2c_1 + 2c_2 = 2 \\ -3c_1 + 2c_2 = 0 \\ \hline +5c_1 = 2 \end{array}$$

$c_1 = 2/5$

$c_1 + c_2 = 1$ $\qquad c_2 = 1 - 2/5 = \underline{3/5}$

$2/5 + c_2 = 1$

$c_2 = 2/5$

$\phi(x) = 3/5\, e^{2x} + 2/5\, e^{-3x}$

5) compute the soln ψ satisfying $\psi(0)=0$, $\psi'(0)=1$

$\phi(x) = c_1 e^{2x} + c_2 e^{-3x}$

$\psi'(x) = c_1 2e^{2x} + c_2 \cdot -3 \cdot e^{-3x}$.

$= 2c_1 e^{2x} - 3c_2 e^{-3x}$

$\psi(0) = c_1 e^0 + c_2 e^0$

$= c_1 + c_2 \quad — ③$

$\psi'(0) = 2c_1 e^0 - 3c_2 e^0$

$= 2c_1 - 3c_2$

$\psi'(0) = 1 \quad = 2c_1 - 3c_2 = 1 \quad — ④$

$③ \times 3 \Rightarrow 3c_1 + 3c_2 = 0$

$2c_1 - 3c_2 = 1$

$5c_1 = 1$

$c_1 = 1/5$

Sub. $c_1 = \frac{1}{5}$ in eqn ③

$c_1 + c_2 = 0$

$1/5 + c_2 = 0$

$c_2 = -1/5$

$\psi(x) = 1/5\, e^{2x} - 1/5\, e^{-3x}$

$$\phi(x) = c_1 e^{2x} + c_2 e^{-3x} = 3/5\, e^{2x} + 2/5\, e^{-3x}$$

$$\phi(1) = 3/5\, e^{0} + 2/5\, e^{-3}$$

$$= \underline{\underline{3/5 + 2/5\, e^{-3}}}$$

$$\psi(x) = 1/5\, e^{2x} - 1/5\, e^{-3x}$$

$$\psi(1) = \underline{\underline{1/5\, e^{2} - 1/5\, e^{-3}}}$$

3 Find all solution ϕ of $y'' + y = 0$ Satisfying

a) $\phi(0) = 1$, $\phi(\pi/2) = 2$

b) $\phi(0) = 0$, $\phi(\pi) = 0$

c) $\phi(0) = 0$, $\phi'(\pi/2) = 0$

d) $\phi(0) = 0$, $\phi(\pi/2) = 0$

a) The eqn is $y'' + y = 0$

$\therefore$ The characteristic polynomial is

$$r^2 + 1 = 0$$

$$r^2 = -1 = i^2$$

$$r = \pm i$$

$$\phi(x) = c_1 \cos x + c_2 \sin x$$

$\phi(0) = 1$, $\phi(\pi/2) = 2$

$$\phi(0) = 1 \Rightarrow c_1 . \cos 0 + c_2 . 0 = 1$$

$$\Rightarrow \underline{\underline{c_1 = 1}}$$

$$\phi(\pi/2) = c_1 . \cos \pi/2 + c_2 \sin \pi/2$$

$$= 0 + c_2 = 2$$

$$\underline{c_2 = 2}$$

$\therefore \phi(x) = \cos x + 2 \sin x$

b) $\phi(0) = 0, \quad \phi(\pi) = 0$

$\phi(0) = c_1 + 0 = 0$

$\therefore c_1 = 0$

$\phi(\pi) = -c_1 + 0 = 0$

$c_1 = 0$

$\therefore \phi(x) = c \sin x$

c) $\phi(0) = 0, \quad \phi'(\pi/2) = 0$

$\phi(x) = c_1 \cos x + c_2 \sin x$

$\phi'(x) = -c_1 \sin x + c_2 \cos x$

$\phi(0) = c_1 + 0 = 0$

$\Rightarrow c_1 = 0$

$\phi'(\pi/2) = -c_1 = 0$

$\phi(x) = c \sin x$

d) $\phi(0) = 0, \quad \phi(\pi/2) = 0$

$\phi(0) = c_1 + 0 = 0$

$\phi(\pi/2) = 0 + c_2 = 0$

$\therefore c_1 = 0, \quad c_2 = 0$

$\therefore \phi(x) = 0$ for all x

4. Consider the equation $y'' + a_1 y' + a_2 y = 0$ where the constant a_1, a_2 are real. $\alpha + i\beta$ is a complex roots of a characteristic polynomial when α, β are real, $\beta \neq 0$ show that $\alpha - i\beta$ is also a root.

show that any solution ϕ may be written in the form $\phi(x) = e^{\alpha x}(d_1 \cos\beta x + d_2 \cos\beta x)$, where d_1, d_2 are constants

Sln From ⓐ and assumption $\alpha + i\beta$ and $\alpha - i\beta$ are Roots of ① ∴ $\phi(x) = c_1 \phi_1(x) + c_2 \phi_2(x)$

$= c_1 e^{(\alpha + i\beta x)} + c_2 e^{(\alpha - i\beta)x}$

$= c_1 e^{\alpha x} e^{i\beta x} + c_2 e^{\alpha x} \cdot e^{-i\beta x}$

$= c_1 e^{\alpha x}(\cos\beta x + \sin\beta x) + c_2 e^{\alpha x}(\cos\beta x - \sin\beta x)$

$= \cos\beta x [c_1 e^{\alpha x} + c_2 e^{\alpha x} + e^{\alpha x} \sin\beta x(c_1 - c_2)]$

$= e^{\alpha x}[d_1 \cos\beta x + d_2 \sin\beta x]$

where $d_1 = c_1 + c_2$, $d_2 = c_1 - c_2$

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d) show that $\alpha = -a_1/2$, $\beta^2 = a_2 - \frac{a_1^2}{4}$

Sln The characteristic poly. $p(r) = 0$.

$p(2\alpha + a_1) = 0$

$\Rightarrow 2\alpha + a_1 = 0$

$\Rightarrow \alpha = -a_1/2$ ∵ $\beta \neq 0$

The second order linear eqn in α is

$\alpha^2 - \beta^2 + a_1 \alpha + a_2 = 0$.

$\left(-\frac{a_1}{2}\right)^2 - \beta^2 + a_1\left(-\frac{a_1}{2}\right) + a_2 = 0$

$\frac{a_1^2}{4} - \beta^2 - \frac{a_1^2}{2} + a_2 = 0$

$$\beta^2 = -\frac{a_1^2}{4} + a_2$$

$$\beta^2 = a_2 - \frac{a_1^2}{4}$$

Show that every solution tends to zero as $x \to \infty$ if $a_1 > 0$

S/n $\alpha = -\frac{a_1}{2} \Rightarrow e^{-a_1/2\ \alpha} = \frac{1}{e^{a_1/2\,\alpha}}$

When $x \to \infty$. and $a_1 > 0$

$$e^{\alpha x} = \frac{1}{e^{\frac{a_1}{2}\alpha}}$$

$$\frac{1}{e^{\alpha}} = \frac{1}{\alpha} = 0 \quad \therefore a_1, a_2 > 0$$

as $x \to \infty$ $\phi(x) \to 0$

Show that the magnitude of every non-trivial soln assume arbitrarily large values as $x \to +\infty$ if $a_1 < 0$

Sln $a_1 < 0$, $e^{\alpha x} = e^{-a_1/2\ x}$

$\left[\because a_1 < 0 \text{ If } x \to \alpha \quad e^{-(a_1/2)\alpha} = e^{-\frac{a_1}{2}\alpha} = e^{\infty}\right.$

$$= e^{\infty} = \infty$$

As $a < 0$ & $x \to \infty$ $\phi(x) \to \infty$ arbitrarily large

Consider the equation

$$Ly'' + Ry' + \frac{1}{c}y = 0$$

where L, R and c are +ve constants

Compute all solution for the three case

$$\frac{R^2}{L^2} - \frac{4}{Lc} > 0$$

Given $Ly'' + Ry' + \left(\frac{1}{c}\right)y = 0$

$$\frac{R^2}{L^2} > \frac{4}{Lc}$$

The characteristic polynomial is

$$p(r) = r^2 L + rR + \frac{1}{c} = 0$$

$$r = \frac{-R \pm \sqrt{R^2 - 4L(1/c)}}{2L}$$

$$= -\frac{R}{2L} \pm \frac{\sqrt{R^2 - 4L/c}}{2L}$$

$$= \frac{-R}{2L} \pm 1/2 \sqrt{\frac{R^2}{L^2} - \frac{4L}{L^2c}}$$

$$= \frac{-R}{2L} \pm \frac{1}{2}\sqrt{\frac{R^2}{L^2} - \frac{4}{Lc}}$$

$$\phi(x) = e^{-R/2L\,x} \left[c_1 \exp\left[-\frac{R}{2L} + \frac{1}{2}\sqrt{\frac{R^2}{L^2} - \cancel{4}4/Lc}\right]x + c_2 \exp\left[\frac{-R}{2L} - \frac{1}{2}\sqrt{\frac{R^2}{L^2} - \frac{4}{Lc}}\right]x\right]$$

(a) $\frac{R^2}{L^2} - \frac{4}{LC} = 0$

$$\frac{R^2}{L^2} = \frac{4}{Lc}$$

$$r = -\frac{R}{2L} \pm \frac{1}{2}\sqrt{\frac{R^2}{L^2} - \frac{R^2}{L^2}}$$

$$= \frac{-R}{2L}$$

$$\phi(x) = (c_1 + c_2 x)\, e\,xp\left(\frac{-R}{2L}\right)x$$

iii) $\dfrac{R^2}{L^2} - \dfrac{4}{LC} < 0$

$$\frac{R^2}{L^2} < \frac{4}{LC}$$

$$r = -\frac{R}{2L} \pm \frac{1}{2}\sqrt{-\left(\frac{4}{LC} - \frac{R^2}{L^2}\right)}$$

$$= \frac{-R}{2L} \pm \frac{1}{2}\left(\frac{4}{LC} - \frac{R^2}{L^2}\right)^{1/2}$$

$$\phi(x) = e^{-\frac{R}{2L}x}\left[c_1 \cos\frac{1}{2}\left(\frac{4}{LC} - \frac{R^2}{L^2}\right)^{1/2} x + c_2 \sin\frac{1}{2}\left[\frac{4}{LC} - \frac{R^2}{L^2}\right]^{1/2} x\right]$$

b) Show that all soln tend to zero as $x \to \infty$ for each of case i), ii), iii) of (a)

case (i) in (a)

$$\phi(x) = \frac{1}{e^{R/2L\, x}}\left[c_1 \exp\left[\frac{-R}{2L} + 1/2\sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}\right)\alpha + c_2 \exp\left[\frac{-R}{2L} - \frac{1}{2}\sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}\right)\alpha\right]$$

solution tends to zero as $x \to \infty$

$$= \frac{1}{e^{R/2L\, \infty}}\left[c_1 \exp\left[-\frac{R}{2L} + \frac{1}{2}\sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}\right]\infty + c_2 \exp\left[\frac{-R}{2L} - \frac{1}{2}\sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}\right]\infty\right.$$

$$= \frac{1}{\infty}\left[c_1 \exp\infty + c_2 \exp\infty\right]$$

$$= 0\ (\infty) = 0$$

case (ii) in a

$$\phi(x) = (c_1 + c_2 x)\frac{1}{e^{R/2L\, x}}$$

Solution tends to zero as $x \to \infty$

$$\left(C_1 + C_2 x\right)\frac{1}{\infty} = \left(C_1 + C_2 x\right) \cdot 0 = 0$$

Case (iii) in ⓐ

$$\phi(x) = \frac{1}{e^{R/2L\,\infty}}\left[C_1 \cos \tfrac{1}{2}\left(\frac{4}{LC} - \frac{R^2}{L^2}\right)^{1/2}\infty + C_2 \sin \tfrac{1}{2}\left[\frac{4}{LC} - \frac{R^2}{L^2}\right]^{1/2}\alpha\right]$$

$$= \frac{1}{\infty}\left(C_1 \cos\infty + C_2 \sin\alpha\right)$$

$$= 0$$

∴ all solns tends to zero as $x \to \infty$
for each case (i), ii) (iii) in ⓐ

c) sketch the sln ϕ satisfying $\phi(0)=1$, $\phi'(0)=0$ in case(iii)

Case iii

$$\phi(x) = e^{-R/2L\,x}\left[C_1 \cos \tfrac{1}{2}\left[\frac{4}{LC} - \frac{R^2}{L^2}\right)^{1/2} x\right] + C_2 \sin \tfrac{1}{2}\left[\frac{4}{LC} - \frac{R^2}{L^2}\right)^{1/2} x\right]$$

$\phi(0) = 1$

$$\phi(0) = e^{-R/2L\,(0)}\left[C_1 \cos \tfrac{1}{2}\left[\frac{4}{LC} - \frac{R^2}{L^2}\right]^{1/2}(0) + C_2 \sin \tfrac{1}{2}\left[\frac{4}{LC} - \frac{R^2}{L^2}\right]^{1/2}(0)\right]$$

$$\therefore 1 = C_1$$

$$\phi'(x) = -\frac{R}{2L}\, e^{-R/2L\,x}\left[-C_1 \sin \tfrac{1}{2}\left[\frac{4}{LC} - \frac{R^2}{L^2}\right)^{1/2} x \;\; \tfrac{1}{2}\left(\frac{4}{LC} - \frac{R^2}{L^2}\right)^{1/2} + C_2 \cos \tfrac{1}{2}\left(\frac{4}{LC} - \frac{R^2}{L^2}\right)^{1/2} \tfrac{1}{2}\left(\frac{4}{LC} - \frac{R^2}{L^2}\right)^{1/2}\right)$$

$$\phi'(0) = -\frac{R}{2L}\,\tfrac{1}{2}\left[\frac{4}{LC} - \frac{R^2}{L^2}\right)^{1/2} C_2$$

$$0 = C_2\left(-\frac{R}{2L}\right)\tfrac{1}{2}\left(\frac{4}{LC} - \frac{R^2}{L^2}\right)^{1/2}$$

$$C_2 = 0$$

$$\phi(x) = e^{-R/2L\, x}\left[\cos\frac{1}{2}\left(\frac{4}{LC}-\frac{R^2}{L^2}\right)^{1/2} x\right]$$

Show that any soln ϕ in case (iii) may be written in the form

$$\phi(x) = A e^{\alpha x} \cos(\beta x - \omega)$$

where A, α, β, ω are constants. Determine α, β

$$\phi(x) = A e^{\alpha x}(\cos(\beta x - \omega))$$

$$= A e^{\alpha x}\left[\cos\beta x \cos\omega + \sin\beta x \sin\omega\right]$$

where A, α, β, ω are constant

If $\alpha = \frac{-R}{2L}$ & $\beta = \frac{1}{2}\left[\frac{4}{LC} - \frac{R^2}{L^2}\right]^{1/2}$

$$\phi(x) = A e^{-R/2L}\left[\cos\frac{1}{2}\left(\frac{4}{LC}-\frac{R^2}{L^2}\right)^{1/2}\cos\omega + \sin\frac{1}{2}\left[\frac{4}{LC}-\frac{R^2}{L^2}\right]^{1/2}\sin\omega\right]$$

Here A, α, β, ω are constant. This similar to case iii) in (a)

6 Show that every soln of the constant coefficient eqn $y'' + a_1 y' + a_2 y = 0$ tend to zero as $x \to \infty$ if and only if, the real parts of the roots of characteristic polynomial are negative

Given $y'' + a_1 y' + a_2 y = 0$

The characteristic poly is $p(r) = r^2 + a_1 r + a_2$

Here $a = 1$, $b = a_1$, $c = a_2$

$$r = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$$

If the roots r_1 and r_2 are real and negative

then $\lim_{x \to \infty} p(x) = \lim_{x\to\infty} C_1 \exp\left(\frac{-a_1 + \sqrt{a_1^2 - 4a_2}}{2}\right) x$

$$+ \lim_{x\to\infty} C_2 \exp\left(\frac{-a_1 - \sqrt{a_1^2 - 4a_2}}{2}\right]$$

$= 0$

$$r = \frac{-a_1 \pm i\sqrt{4a_2 - a_1^2}}{2}$$

If the roots are complex i.e. $r_1 = \frac{-a_1 + i\sqrt{4a_2 - a_1^2}}{2}$

$r_2 = \frac{-a_1 - i\sqrt{4a_2 - a_1^2}}{2}$ with $-\frac{a_1}{2} < 0$

then

$\lim_{x\to\infty} p(x) = \lim_{x\to\infty} e^{-a_1/2\, x}\left[A \cos \frac{\sqrt{4a_2 - a_1^2}}{2} x + B \sin \frac{\sqrt{4a_2 - a_1^2}}{2} x\right.$

$= 0$

In this case the solutions are often called transients

Transient - when the limit $x \to \infty$ tends to zero
that response that approaches to zero

Consider the eqn $y'' + k^2 y = 0$, where k is a nonnegative constant

a) For what values of k will there exis non trivial solns ϕ satisfying

I) $\phi(0) = 0$, $\phi(\pi) = 0$

b) $\phi'(0) = 0$, $\phi'(\pi) = 0$

(III) $\phi(0) = \phi(\pi)$, $\phi'(0) = \phi'(\pi)$)

iv) $\phi(0) = -\phi(\pi)$, $\phi'(0) = -\phi'(\pi)$

b) Find the non trivial solutions of each of the case (i) – (v) in (a)

$$y'' + k^2 y = 0$$

The characteristic polynomial is

$$p(r) = r^2 + k^2$$

$$r^2 = -k^2$$

$$r = \sqrt{-k^2} = \underline{\pm ki}$$

Therefore any soln of ϕ has the form

$$\phi(x) = c_1 \cos kx + c_2 \sin kx.$$

i) $\phi(0) = 0$, $\phi(\pi) = 0$

$$\phi(0) = c_1 \cos 0 + c_2 \sin 0$$

$$0 = c_1, \quad c_1 = 0$$

$$\phi(\pi) = 0$$

$$\phi(\pi) = c_1 \cos \pi + c_2 \sin \pi$$

$$\underline{0 = c_1}, \quad \underline{c_1 = 0}$$

$$\phi(x) = c \sin kx, \quad k = 1, 2, 3 \ldots$$

(ii) $\phi'(0) = 0$, $\phi'(\pi) = 0$

$$\phi'(x) = -c_1 \sin kx + c_2 \cos kx$$

$$\phi'(0) = k c_2$$

$$\therefore \underline{c_2 = 0}$$

$$\phi'(\pi) = -c_1 \sin \pi x \, k + c_2 \cos \pi x \, k$$

$$c_2 = 0$$

$$\phi(x) = c \cos kx, \quad k = 0, 1, 2, 3.$$

(iii) $\phi(0) = \phi(\pi)$, $\phi'(0) = \phi(\pi)$

$\phi(0) = c_1 \cos 0 + c_2 \sin 0$

$c_1 = 0 \Rightarrow \underline{c_1 = 0}$

$\phi(\pi) = c_1 \cos \pi + c_2 \sin \pi$

$0 = -c_1 \Rightarrow \underline{c_1 = 0}$

$\phi(x) = c_2 \sin kx$

$\phi'(\pi) = -c_1 \sin kx \cdot k + c_2 \cos kx \cdot k$

$= -c_1 \sin \pi k \cdot k + c_2 \cos \pi k \cdot k$

$0 = -k c_2$

$c_2 = 0$

$\phi'(0) = -c_1 \sin 0 x k + c_2 \cos 0 x \cdot k$

$0 = k c_2$

$c_2 = 0$

$\phi(x) = c_1 \cos kx$

$\phi(x) = \underline{c_1 \cos kx + c_2 \sin kx}$

iv) $\phi(0) = -\phi(\pi)$, $\phi'(0) = -\phi'(\pi)$

$\phi(0) = c_1 \cos 0 + c_2 \sin 0$

$\phi(0) = c_1$

$-\phi(\pi) = -[c_1 \cos \pi + c_2 \sin \pi)$ $\Big]$ $\phi(x) = c_2 \sin kx$

$-\phi(\pi) = c$

$\phi'(0) = c_2 = 0$

$-\phi'(\pi) \Rightarrow c_2 = 0$ $\Big\}$ $\phi(x) = c_1 \cos kx$

$\phi(x) = c_1 \cos kx + c_2 \sin kx$

$\phi(x) = c_1 \cos(2n-1)x + c_2 \sin(2n-1)x$

For $k = 2n-1$, $n = 1, 2, 3$

Let ϕ be a soln of the equation

$y'' + a_1 y' + a_2 y = 0$

where a_1, a_2 are constants. If

$\psi(x) = e^{(a_1/2)(x)} \phi(x)$

Show that ψ satisfies an equation $y'' + ky = 0$

where k is a constant. Compute k

$y'' + a_1 y' + a_2 y = 0$

The characteristic polynomial is

$p(r) = r^2 + a_1 r + a_2$

$p(r) = 0 \Rightarrow$

$r^2 + a_1 r + a_2 = 0$

$\therefore r = \dfrac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$

$= -\dfrac{a_1}{2} \pm \dfrac{\sqrt{a_1^2 - 4a_2}}{2}$

$\phi(x) = e^{-a_1/2\, x}\left[c_1 \cos \dfrac{\sqrt{4a_2 - a_1^2}}{2} + c_2 \sin \dfrac{\sqrt{4a_2 - a_1^2}}{2}\right.$

$\psi(x) = e^{(a_1/2)x} \phi(x)$

$\psi(x) = c_1 \cos \dfrac{\sqrt{4a_2 - a_1^2}}{2} x + c_2 \sin \dfrac{\sqrt{4a_2 - a_1^2}}{2}$

$y'' + ky = 0$

$p(r) = r^2 + k = 0$

$r^2 = -k \Rightarrow r = \pm i k$

$\psi(x) = e^{0x}\left[c_1 \cos kx + c_2 \sin kx\right\}$

where $k = \frac{\sqrt{4a_2 - a_1^2}}{2}$

$\phi(x) = C_1 \cos\left(\frac{\sqrt{4a_2 - a_1^2}}{2}\right)x + C_2 \sin\left(\frac{\sqrt{4a_2-a_1^2}}{2}\right)(x)$

1. Find all the solution of the following initial value problem

a) $y'' - 2y' - 3y = 0$, $y(0) = 0$, $y'(0) = 1$

Sln Let $L(y) = y'' - 2y' - 3y$

An initial value problem for $L(y) = 0$ to find a solution of ϕ

$y'' - 2y' - 3y = 0$

The characteristic polynomial is

$p(r) = r^2 - 2r - 3$

$r^2 - 2r - 3 = 0$

$r = 3, -1$

$\phi(x) = C_1 \cos 3x + C_2 e^{-x}$

$y(0) = 0$

$\phi(0) = C_1 e^0 + C_2 e^0$

$C_1 + C_2 = 0$ — (1)

$\phi'(x) = C_1 e^{3x} \cdot 3 + C_2 e^{-x}(-1)$

$y'(0) = 1 \Rightarrow 3C_1 - C_2 = 1$ — (2)

$c_1 + c_2 = 0$

$3c_1 - c_2 = 1$

$4c_1 = 1 \Rightarrow c_1 = 1/4$

Sub. c_1 in ①

we get $1/4 + c_2 = 0 \Rightarrow c_2 = -1/4$

$\therefore \phi(x) = 1/4\, e^{3x} - 1/4\, e^{-x}$

6) $y'' + (4i+1)y' + y = 0$, $y(0) = 0$, $y'(0) = 0$

Given $y'' + (4i+1)y' + y = 0$

The characteristic poly is

$$p(r) = r^2 + (4i+1)r + 1$$

$$r^2 + (4i+1)r + 1 = 0$$

$$r = \frac{-(4i+1) \pm \sqrt{(4i+1)^2 - 4}}{2}$$

$$= \frac{-4i - 1 \pm \sqrt{(4i+1)^2 - 2^2}}{2}$$

$$= \frac{-4i - 1 \pm 4i + 1 - 2}{2}, \quad \frac{-4i - 1 - (4i + 1 - 2)}{2}$$

$$= \frac{-4i - 1 + 4i + 1 - 2}{2}, \quad \frac{-4i - 1 - 4i - 1 + 2}{2}$$

$$= -\frac{2}{2} \quad \& \quad -\frac{8i}{2}$$

$$-1, -4i$$

$$\phi(x) = c_1 e^{-x} + c_2 e^{-4ix}$$

$y(0)=0$

$\phi(0) = c_1 + c_2$

$c_1 + c_2 = 0$ — (1)

$y'(0) = 0$

$\phi'(x) = c_2 e^{-x}(-1) + c_2 e^{-4ix}(-4i)$

$\phi'(0) = c_1 e^{0}(-1) + c_2 e^{-4.0}(-4i)$

$-c_1 - 4i c_2 = 0$ — (2)

$$\begin{array}{l} c_1 + c_2 = 0 \\ -c_1 - 4i c_2 = 0 \\ \hline c_2 - 4i c_2 = 0 \end{array}$$

$c_2 (1 - 4i) = 0$

$c_2 = 0$

$c_1 + c_2 = 0$

$c_1 + 0 = 0$

$c_1 = 0$

$\phi(x) = 0 \cdot e^{-x} + 0 \cdot e^{-4ix}$

$\phi(x) = 0$ for all x

c. $y'' + (3i - 1) y' - 3i y = 0$

$y'' + (3i-1) y' - 3i y = 0$

The characteristic poly. $p(r) = r^2 + (3i-1) r - 3i$

ie, $r^2 + (3i - 1) r - 3i = 0$

$$r = \frac{-(3i-1) \pm \sqrt{(3i-1)^2 - 4 \cdot 3i}}{}$$

$$\frac{-(3i-1) \pm \sqrt{9i^2-6i+1+12i}}{2}$$

$$= \frac{-3i+1 \pm \sqrt{9i^2-6i+1}}{2}$$

$$= \frac{-3i+1 \pm 3i+1}{2}$$

$$= \frac{-3i+1+3i+1}{2}, \frac{-3i+1-3i-1}{2}$$

$$= -\frac{6i}{2}, \frac{2}{2}$$

$$\Rightarrow 3i, 1$$

$$\phi(x) = c_1 e^{x} + c_2 e^{-3ix}$$

$y(0) = 0$

$$2 = c_1 + c_2 \text{ — } (1)$$

$y'(0) = 0$

$$\phi'(x) = c_1 e^{x} - 3i c_2 e^{-3ix}$$

$$0 = c_1 - 3i c_2 \rightarrow (2)$$

$$c_1 + c_2 = 2$$

$$c_1 + -3i c_2 = 0$$

$$c_2 + 3i c_2 = 2$$

$$c_2(1+3i) = 2$$

$$c_2 = \frac{2}{1+3i}$$

$$= \frac{2}{3i+1} \times \frac{3i-1}{3i-1} = \frac{2(3i-1)}{-9-1}$$

$$= \frac{2(3i-1)}{-10}$$

Sub c_2 in (1)

$$c_1 + \frac{1-3i}{5} = 2$$

$$c_1 = \frac{10}{1-3i} \times \frac{1+3i}{1+3i}$$

$$= 2 - \frac{1+3i}{5}$$

$$= \frac{10 - 1 + 3i}{5} = \frac{9+3i}{5}$$

$$\phi(x) = \frac{3}{5}(3+i)\, e^{x} + \frac{1}{5}(1+3i)\, e^{-3ix}$$

d) $y'' + 10y = 0, \quad y(0) = \pi, \quad y'(0) = \pi^2$

Given

$$y'' + 10y = 0$$

The characteristic poly $p(r) = r^2 + 10$

$p(r) = 0 \Rightarrow r^2 + 10 = 0$

$$r^2 = -10, r = \pm i\sqrt{10}$$

$$\phi(x) = e^{0x}\left[c_1 \cos\sqrt{10}\,x + c_2 \sin\sqrt{10}\,x\right]$$

$y(0) = \pi \Rightarrow \pi = c_1$

$\phi'(x) = c_1 \sin\sqrt{10} \cdot \sqrt{10} + c_2 \cos\sqrt{10} \cdot \sqrt{10}$

$y'(0) = \pi^2$

$$\pi^2 = c_2\sqrt{10}$$

$$\frac{\pi^2}{\sqrt{10}} = c_2$$

$$\phi(x) = \pi \cos\sqrt{10}\, x + \frac{\pi^2}{\sqrt{10}} \sin\sqrt{10}\, x$$

Find a function ϕ which has a continuous derivative on $0 \le x \le$ which satisfies

$$\phi(0) = 0, \quad \phi'(0) = 1$$

and $y'' - y = 0$ for $0 \le x \le 1$

and $y'' - 9y = 0$ for $1 \le x \le 2$

Sol: $y'' - y = 0$

$$p(r) = r^2 - 1 = 0$$

$$r^2 = 1 \Rightarrow \underline{r = \pm 1}$$

$$\phi(x) = c_1 e^x + c_2 e^{-x}, \quad \underline{\phi'(x) = c_1 e^x - c_2 e^{-x}}$$

$$\phi(0) = c_1 e^0 + c_2 e^{-0} = c_1 + c_2 = 0$$

$$\phi'(0) = c_1 - c_2 = 1$$

$$\begin{aligned} c_1 + c_2 &= 0 \\ c_1 - c_2 &= 1 \\ \hline 2c_1 &= 1 \end{aligned} \qquad c_1 = 1/2$$

$$c_2 = -1/2$$

$$\phi(x) = \frac{e^x - e^{-x}}{2}$$

$$L(y) = y'' - 9y = 0.$$

$$p(r) = r^2 - 9 = 0$$

$$r^2 = 9$$

$r = \pm 3$

$\phi(x) = c_1 e^{3x} + c_2 e^{-3x}$

$\phi'(x) = 3c_1 e^{3x} - 3c_2 e^{-3x}$

$\phi(0) = c_1 + c_2 = 0 \quad - (1)$

$\phi'(0) = 3c_1 - 3c_2 = 1 \quad - (2)$

$(3) x + (2) \Rightarrow 6c_1 = 1$

$c_1 = 1/4$

$c_2 = -1/6$

$\phi(x) = 1/6 \, (e^{3x} - e^{-3x})$